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A NEW TYPE OF SOLUTION OF MAXWELL'S EQUATIONS.

BY H. BATEMAN.

1. There is some experimental evidence of the existence of radiations which are apparently corpuscular in character but also possess the properties of waves or pulses travelling through a continuous medium. The observed phenomena can probably be explained satisfactorily by supposing that the effective part of the disturbance is concentrated in certain regions and almost absent in others. It may, however, be of some interest to inquire if there are any solutions of Maxwell's equations which correspond to a corpuscular type of radiation. The solution which is discussed here fulfils this requirement but is of a type which is generally rejected in the treatment of the boundary problems of mathematical physics. It may not, therefore, correspond to any physical reality. Nevertheless, I have thought it worth while to place the solution on record as it furnishes a good example of the theory of the mutual connection of electromagnetic fields and transformations which can be applied to them. A theory which I have developed in some previous papers.

2. Consider a point M moving along a curve Γ with a velocity v less than c the velocity of light. We may represent its position at time τ by the equations

$$(1) \qquad x = \xi(\tau), \qquad y = \eta(\tau), \qquad z = \zeta(\tau).$$

Associated with any space-time point (x, y, z, t) there is just *one** time τ such that if $t \geq \tau$

$$(2) \qquad [x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 + [z - \zeta(\tau)]^2 = c^2(t - \tau)^2.$$

The corresponding position of M is called its *effective position* relative to (x, y, z, t).

Now let $l(\tau)$, $m(\tau)$, $n(\tau)$, $p(\tau)$ be functions connected by the equations

$$(3) \quad l(\tau)\xi'(\tau) + m(\tau)\eta'(\tau) + n(\tau)\zeta'(\tau) = c^2p(\tau), \qquad l^2 + m^2 + n^2 = c^2p^2,$$

and write

$$(4) \qquad \sigma = l(x - \xi) + m(y - \eta) + n(z - \zeta) - c^2p(t - \tau).$$

It is easy to verify by a slight modification of the method used in a former paper* that the function

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* A. Liénard, L'éclairage électrique, vol. 16 (1898), pp. 5, 53, 106; A. W. Conway, Proc. London Math. Soc., Ser. 2, vol. 1, p. 154 (1903); H. Minkowski, Raum und Zeit, Phys. Zeitschr. (1909); H. Bateman, Manchester Memoirs (1910). See also G. A. Schott's Adams Prize Essay, 'Electromagnetic Radiation,' Cambr. Univ. Press (1912).

$$V = \frac{f(\tau)}{\sigma}$$

is a solution of the wave equation

$$\Omega(v) \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0.$$

Moreover, if we derive an electromagnetic field from the potentials

$$(5) \qquad A_x = \frac{l}{\sigma}, \qquad A_y = \frac{m}{\sigma}, \qquad A_z = \frac{n}{\sigma}, \qquad \Phi = \frac{p}{\sigma},$$

we find that the equation

$$\operatorname{div} A + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0.$$

is satisfied, and that there is no charge or convection current at space-time points for which $\sigma \neq 0$.

The quantity σ vanishes when (x, y, z, t) is on Γ and also when it lies on one of two lines through the effective position of M. To see this let (ξ', η', ζ') be regarded as the coördinates of a point U within the sphere

$$X^2 + Y^2 + Z^2 = c^2$$

which we shall denote by S. The conditions (3) imply that the point Q whose coördinates are $(l/cp, m/cp, n/cp)$ lies on the polar plane of U and also on the sphere. Now the polar plane of U lies outside the sphere and so only meets it in imaginary points, hence the ratios l/p, m/p, n/p are not all real. Since, moreover, the polar plane of U meets the sphere S in an imaginary circle C it is clear that there are ∞' possible sets of values of l, m, n, p. Now let $(x - \xi)/c(t - \tau)$, $(y - \eta)/c(t - \tau)$, $(z - \zeta)/c(t - \tau)$ be regarded as the coördinates of a point P on the sphere S. The tangent plane at P meets the circle C in two imaginary points R, R'. If one of these coincides with the point Q the expression σ vanishes.

Now if $Q \equiv R$ the coördinates of R' can be written down at once by changing i into $-i$ in the coördinates of R. Having found R' we can determine P by drawing tangent planes through RR' to the sphere. The points of contact of these two planes will be the two possible positions of P. Now these points lie on the polar line of RR' and this is a line through U perpendicular to RR' which cuts the sphere in two *real* points PP'.

Taking the real (or i times the imaginary) parts of the potentials (5) we obtain a real electromagnetic field with the following properties. *There is a primary singularity M moving with a velocity less than that of light along*

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* *Annals of Mathematics*, 2d series, vol. 14, Dec. (1912).

a curve Γ. *There are secondary singularities radiating with the velocity of light from points of the curve* Γ, *there being two rectilinear rays through each point of this curve.* The direction of motion of M bisects the angle between the two rays and the angle which it makes with either has a cosine equal to v/c, where v is the velocity of M.

It should be noticed that if M is moving along a straight line with constant velocity and l, m, n, p are regarded as constants, the electric and magnetic forces derived from the potentials (5) are everywhere null. *Hence the electromagnetic field under consideration is of a type which only comes into existence when the velocity of the primary singularity changes*, or the directions of the rays alter.

The components of the electric and magnetic vectors D, H are given by equations of type

$$H_x \equiv \frac{\partial}{\partial y}\left(\frac{n}{\sigma}\right) - \frac{\partial}{\partial z}\left(\frac{m}{\sigma}\right) = \left(\frac{n'}{\sigma} - \frac{Kn}{\sigma^2}\right)\frac{\partial\tau}{\partial y} - \left(\frac{m'}{\sigma} - \frac{Km}{\sigma^2}\right)\frac{\partial\tau}{dz},$$

$$D_x \equiv -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{l}{\sigma}\right) - \frac{\partial}{\partial x}\left(\frac{cp}{\sigma}\right) = -\frac{1}{c}\left(\frac{l'}{\sigma} - \frac{Kl}{\sigma^2}\right)\frac{\partial\tau}{\partial t} - c\left(\frac{p'}{\sigma} - \frac{Kp}{\sigma^2}\right)\frac{\partial\tau}{\partial x},$$

where

$$K \equiv l'(x - \xi) + m'(y - \eta) + n'(z - \zeta) - c^2p'(t - \tau).$$

It follows from these equations that*

$$\begin{aligned}
&\frac{\partial\tau}{\partial x}H_x + \frac{\partial\tau}{\partial y}H_y + \frac{\partial\tau}{\partial z}H_z = 0,\\
&\frac{1}{c}\frac{\partial\tau}{\partial t}H_x + \frac{\partial\tau}{\partial y}D_z - \frac{\partial\tau}{\partial z}D_y = 0,\\
(6)\qquad &\\
&\frac{\partial\tau}{\partial x}D_x + \frac{\partial\tau}{\partial y}D_y + \frac{\partial\tau}{\partial z}D_z = 0,\\
&\frac{1}{c}\frac{\partial\tau}{\partial t}D_x - \frac{\partial\tau}{\partial y}H_z + \frac{\partial\tau}{\partial z}H_y = 0.
\end{aligned}$$

Hence the electric and magnetic vectors are at right angles and are equal in magnitude. Poynting's vector has components proportional to $\partial\tau/\partial x$, $\partial\tau/\partial y$, $\partial\tau/\partial z$ *and so is along the radius from the effective position of* M.

When the real or imaginary parts of D and H only are retained all the above conclusions remain valid.

3. Another conclusion which may be drawn from equations (6) is that the electromagnetic field can be transformed into another one of the same

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* We make use here of the relations

$$\frac{\dfrac{\partial\tau}{\partial x}}{x - \xi} = \frac{\dfrac{\partial\tau}{\partial y}}{y - \eta} = \frac{\dfrac{\partial\tau}{\partial z}}{z - \zeta} = -\frac{\dfrac{\partial\tau}{\partial t}}{c^2(t - \tau)}.$$

general type but associated with a different curve Γ'. The equations of the transformation are*

$$(7)\qquad \begin{aligned} x' &= xf(\tau) - \int_{\tau_0}^{\tau} f'(\tau)\xi(\tau)d\tau, & z' &= zf(\tau) - \int_{\tau_0}^{\tau} f'(\tau)\zeta(\tau)d\tau, \\ y' &= yf(\tau) - \int_{\tau_0}^{\tau} f'(\tau)\eta(\tau)d\tau, & t' &= tf(\tau) - \int_{\tau_0}^{\tau} \tau f'(\tau)d\tau, \end{aligned}$$

where τ is defined in terms of x, y, z, t by the equations (2) and τ_0 is a constant.

This transformation makes the curve Γ correspond to a curve Γ' defined by the equations

$$(8)\qquad x' = \alpha(\tau), \qquad y' = \beta(\tau), \qquad z' = \gamma(\tau), \qquad t' = \theta(\tau),$$

where

$$\begin{aligned} \alpha'(\tau) &= f(\tau)\xi'(\tau), & \beta'(\tau) &= f(\tau)\eta'(\tau), \\ \gamma'(\tau) &= f(\tau)\zeta'(\tau), & \theta'(\tau) &= f(\tau). \end{aligned}$$

It is clear from these equations that the two curves Γ and Γ' have the same spherical indicatrix and that the velocities at corresponding points are equal. If Γ is a plane curve so also is Γ'. The relation

$$(9)\qquad (x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2 = c^2(t' - \theta)^2$$

is evidently a consequence of (2). The inequality $t' \geq \theta$ is a consequence of $t \geq \tau$ if $f(\tau)$ is positive and then θ increases with τ.

Defining the potentials for the corresponding electromagnetic field by the equations

$$(10)\qquad A_x' = \frac{l}{\sigma'}, \qquad A_y' = \frac{m}{\sigma'}, \qquad A_z' = \frac{n}{\sigma'}, \qquad \Phi = \frac{cp}{\sigma'},$$

where

$$\sigma' = l(x' - \alpha) + m(y' - \beta) + n(z' - \gamma) - c^2p(t' - \theta),$$

it is easy to see that $\sigma' = \sigma f(\tau)$ and

$$(11)\quad A_x'dx' + A_y'dy' + A_z'dz' - c\Phi'dt' = A_xdx + A_ydy + A_zdz - c\Phi dt + \frac{f'}{f}d\tau.$$

From this equation we may deduce that†

$$(12)\qquad \begin{aligned} &H_x'd(y', z') + H_y'd(z', x') + H_z'd(x', y') + cD_x'd(x', t') \\ &\qquad + cD_y'd(y', t') + cD_z'd(z', t') = H_xd(y, z) + H_yd(z, x) \\ &\qquad + H_zd(x, y) + cD_xd(x, t) + cD_yd(y, t) + cD_zd(z, t), \end{aligned}$$

and so the relations between the electric and magnetic vectors at corresponding points of the two corresponding fields may be written down at once.

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* Proc. London Math. Soc., Ser. 2, vol. 10 (1911), p. 96.

† The method of derivation is explained in former papers. See the next reference.

The fact that this transformation implies the covariance of the electromagnetic equations is a consequence of equations (6) and (12) as is indicated by the analysis in a former paper.*

4. In conclusion it may be worth while to point out that the solution of Laplace's equation given in my former paper may be obtained by contour integration.

It is known that $z / \sqrt{x + iy} \cdot 1/r^2$ is a solution of Laplace's equation: for it may be written in the form

$$\frac{1}{2\sqrt{x+iy}}\left[\frac{1}{z+i\rho}+\frac{1}{z-i\rho}\right],$$

where $\rho^2 = x^2 + y^2$, and $(1 / \sqrt{x + iy})f(z + i\rho)$ is known to be a solution.

Generalizing this by a change of rectangular axes we may say that

$$\frac{\xi'(u)[x-\xi]+\eta'(u)[y-\eta]+\zeta'(u)[z-\zeta]}{\{l(u)[x-\xi]+m(u)[y-\eta]+n(u)[z-\zeta]\}^{\frac{1}{2}}}\,\frac{1}{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$$

is a solution provided

$$l\xi'(u) + m\eta'(u) + n\zeta'(u) = 0,$$
$$l^2 + m^2 + n^2 = 0.$$

Integrating round a closed contour containing only one root of the equation

$$[x - \xi(u)]^2 + [y - \eta(u)]^2 + [z - \zeta(u)]^2 = 0$$

and no root of the equation

$$l(x - \xi) + m(y - \eta) + n(z - \zeta) = 0$$

we obtain the required result.

Another theorem of a similar character is that if $F(X, Y, Z)$ is a homogeneous function of degree $\frac{1}{2}$ satisfying Laplace's equation, then

$$V = F[x - \xi, y - \eta, z - \zeta]$$

is also a solution of Laplace's equation. The corresponding theorems for the equation of wave motion are as follows:

If τ is defined by equation (2) and $F(X, Y, Z, T)$ is a homogeneous function of degree zero satisfying the equation of wave motion, the function

$$V = F[x - \xi, y - \eta, z - \zeta, t - \tau]$$

also satisfies the equation of wave motion.†

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* Proc. London Math. Soc., Ser. 2, vol. 8, pp. 223, 469.

† Three particular cases of this result are given by E. T. Whittaker, Proc. London Math. Soc., Ser. 2, vol. 1 (1903).

If $F[X, Y, Z, T, \tau]$ is of degree -1 in X, Y, Z, T and is a homogeneous function of these variables satisfying the equation of wave motion and the additional equation

$$\xi' \frac{\partial F}{\partial X} + \eta' \frac{\partial F}{\partial Y} + \zeta' \frac{\partial F}{\partial Z} - \frac{1}{c^2} \frac{\partial F}{\partial T} = 0,$$

then the function

$$V = F[x - \xi(\tau),\ y - \eta(\tau),\ z - \zeta(\tau),\ t - \tau,\ \tau]$$

satisfies the equation of wave motion.

The transformation (7) converts these wave potentials into wave potentials in the (x', y', z', t') coördinates. The new potentials are of the same general character as before but are associated with the curve Γ'.

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